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LAMINAR FLOW AND HEAT TRANSFER OVER A TWO-DIMENSIONAL TRIANGULAR STEP

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SUMMARY

The velocity correction algorithm is used in the finite element method to solve forced convection problems between parallel plates with a triangular step, for Reynolds numbers up to 1000. Equal-order interpolation functions for velocity, pressure and temperature are used. The solutions show a smooth variation of pressure. The streamfunction, isotherms, isobars and velocity profiles are presented for a typical Reynolds number of 500. The skin friction and heat transfer results are presented for Reynolds numbers up to 1000.

KEY WORDS Laminar flow Triangular step FEM Velocity correction

INTRODUCTION

The study of laminar flow through constrictions and of flow over steps of different shapes in conduits is of relevance in the field of heat exchanger design, where the pressure drop and heat transfer performance are important. Different kinds of obstructions on the surfaces may occur as a result of imperfect manufacturing processes used in making conduits. Obstructions may also be introduced deliberately in order to produce turbulence in the flow. The study of the behaviour of such obstructions in flows will be of help in judging the performance of equipment of which they form a part.

Laminar flow is also encountered in the flow of biological fluids, such as blood through veins. The obstructions encountered in such flows are of complex shape. The shape can be idealized to a known, simple shape and the effect of it on blood pressure and flow can be studied.

The parallel plate duct geometry is a limiting geometry for the family of rectangular ducts and also for concentric annular ducts. Detailed analytical results for laminar flow and heat transfer for parallel plates have been given by Shah and London.[1] Flow through constrictions in a parallel channel was studied by Weisbach[2] and Kays.[3] Their studies were of an analytical nature. Kays presented semi-empirical relations for expansion and contraction coefficients. Bunditkul,[4,5]

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Greenspan[6] and Friedman[7] conducted numerical studies of flow through long rectangular constrictions by the finite difference method. Banditkul[4, 5] reported that the heat transfer and pressure drop were significantly enhanced. Hughes[8] solved the problem of flow over short square steps up to a Reynolds number of 200 by an upwind finite element method. Gresho[9] and Gartling[10] contested the use of upwinding and Gresho[9] presented results for flow over a square step up to a Reynolds number of 200 without upwinding. The major problems that are encountered in solving steady or unsteady Navier–Stokes equations are:

(i) the treatment of the incompressibility constraint
(ii) the treatment of convective terms.

Schneider *et al.*,[11] Gresho,[9] Reddy,[12] Hughes,[8] Spalding,[13] Godbole,[14] Heinrich[15] and Gartling[10] suggested methods to overcome these problems. The results were reported only up to a Reynolds number of 200 for flow over a square step. Donea[16] used a fractional step method in a finite element context. He has presented results for a similar square step problem up to a Reynolds number of 200 using the velocity correction method. The velocity correction method does not pose any problems in the treatment of the continuity equation and convective terms. The interpolations of pressure and velocities can be of equal order, unlike the unequal interpolation suggested in some of the references above. It does not require any upwinding. Moreover, it does not produce any spurious oscillations in pressure as reported by Gresho.[17] The references cited above deal only with problem of flow over a square step up to a Reynolds number of 200. Nothing has been stated about heat transfer except by Bunditkul.[4, 5]

The present paper is devoted to the application of the velocity correction method to convective heat transfer problems in parallel channels with and without a step of triangular shape. The flow and heat transfer results are presented up to a Reynolds number of 1000. The velocity correction method used for analysis is presented in a simple form below.

ANALYSIS

The equations governing the unsteady, incompressible, viscous two-dimensional flow with energy can be represented in non-dimensional form as

$$DU/D\tau = -dP/dX + (2/Re)\,\partial^2 U/\partial X^2 + (1/Re)\,\partial(\partial U/\partial Y + \partial V/\partial X)/\partial Y, \tag{1a}$$

$$DV/D\tau = -dP/dY + (1/Re)\,\partial(\partial U/\partial Y + \partial V/\partial X)/\partial X + (2/Re)\,\partial^2 V/\partial Y^2, \tag{1b}$$

$$\partial U/\partial X + \partial V/\partial Y = 0, \tag{1c}$$

$$D\theta/D\tau = (1/Re\,Pr)(\partial^2\theta/\partial X^2 + \partial^2\theta/\partial Y^2), \tag{1d}$$

where the non-dimensional quantities are represented by

$$U = u/u_e, \quad P = (p - p_o)/\rho u_e^2, \quad X = x/D,$$
$$V = v/u_e, \quad \theta = (T - T_e)/(T_w - T_e), \quad Y = y/D,$$
$$\tau = tu_e/D, \quad R = \rho D u_e/\mu, \quad Pr = \mu c/k.$$

Equations (1a)–(1d) can be solved simultaneously for a coupled problem. Equations (1a)–(1c) are solved to get the velocities and pressure in an uncoupled problem. The temperatures are evaluated from (1d) by substitution of the calculated velocities.

U_n, V_n, P_n, θ_n and U_{n+1}, V_{n+1}, P_{n+1}, θ_{n+1} are assumed to be the values of the variables at time τ_n and τ_{n+1} respectively. Let a set of fictitious values of velocities $\underline{U}_n$, $\underline{V}_n$ satisfy the equations

$$(\underline{U}_n - U_n)/\Delta\tau = (2/Re)\partial^2 U_n/\partial X^2 + (1/Re)\partial(\partial U_n/\partial Y + \partial V_n/\partial X)/\partial Y, \tag{2a}$$

$$(\underline{V}_n - V_n)/\Delta\tau = (1/Re)\partial(\partial U_n/\partial Y + \partial V_n/\partial X)/\partial X + (2/Re)\partial^2 V_n/\partial Y^2. \tag{2b}$$

The velocities $\underline{U}_n$, $\underline{V}_n$ will not satisfy the continuity equation (1c). Let the equations

$$(U_{n+1} - U_n)/\Delta\tau = -\mathrm{d}P/\mathrm{d}X + (2/Re)\partial U_n^2/\partial X^2 + (1/Re)\partial(\partial U_n/\partial Y + \partial V_n/\partial X)/\partial Y, \tag{3a}$$

$$(V_{n+1} - V_n)/\Delta\tau = -\mathrm{d}P/\mathrm{d}Y + (1/Re)\partial(\partial U/\partial Y + \partial V/\partial X)/\partial X + (2/Re)\partial^2 V_n/\partial Y^2 \tag{3b}$$

be satisfied at the end of time τ_{n+1}. By rearrangement of equations (2) and (3), the following simpler equations are obtained:

$$U_{n+1} = \underline{U}_n - \mathrm{d}P_{n+1}/\mathrm{d}X, \tag{4a}$$

$$V_{n+1} = \underline{V}_n - \mathrm{d}P_{n+1}/\mathrm{d}Y. \tag{4b}$$

The velocities U_{n+1} and V_{n+1} have to satisfy the continuity equation (1c) for them to be the actual velocities at the time τ_{n+1}. Hence, by taking appropriate derivatives of the equations (4a) and (4b) and substituting in equation (1c), the following Poisson equation in pressure is obtained:

$$\mathrm{d}^2P_{n+1}/\mathrm{d}X^2 + \mathrm{d}^2P_{n+1}/\mathrm{d}Y^2 = \mathrm{d}\underline{U}_n/\mathrm{d}X + \mathrm{d}\underline{V}_n/\mathrm{d}Y. \tag{5}$$

The pressure distribution P_{n+1} can be obtained if the values of $\underline{U}_n$ and $\underline{V}_n$ are known. The algorithm to arrive at the final quantities at time τ_{n+1} is as follows.

(i) The values U_n, V_n, P_n and θ_n are assumed to be known at time τ_n.
(ii) The fictitious values $\underline{U}_n$ and $\underline{V}_n$ are evaluated from (2).
(iii) The pressure P_{n+1} is evaluated from (5) by substituting the known values of $\underline{U}_n$ and $\underline{V}_n$.
(iv) The solution of P_{n+1} is substituted in equation (4) to obtain the final values of U_{n+1} and V_{n+1}.
(v) The values of U_{n+1} and V_{n+1} are substituted in the equation

$$(\theta_{n+1} - \theta_n)/\Delta\tau = -U_{n+1}\partial\theta_n/\partial X - V_{n+1}\partial\theta_n/\partial Y + (1/Re\,Pr)(\partial^2\theta_n/\partial X^2 + \partial^2\theta_n/\partial Y^2), \tag{6}$$

which is obtained by explicit Euler time splitting of equation (1d). The temperature θ_n from the previous iteration is assumed to be known.

The solution domain is discretized into a finite number of elements. The same grid is used for the solution in all of the steps described above. In the present work the domain is divided into a number of linear triangles. The variations of velocities, pressure and temperature are assumed to be linear:

$$U = N_iU_i + N_jU_j + N_kU_k, \tag{7a}$$

$$V = N_iV_i + N_jV_j + N_kV_k, \tag{7b}$$

$$P = N_iP_i + N_jP_j + N_kP_k, \tag{7c}$$

$$\theta = N_i\theta_i + N_j\theta_j + N_k\theta_k. \tag{7d}$$

All the variables are interpolated to equal order. Since linear elements are used, the numerical integrations involved in the evaluation of element matrices are avoided.

NUMERICAL EXAMPLES

The following problems are solved to illustrate the capability of the method:

(i) flow through a channel with a backward step
(ii) the temperature development in flow between parallel plates, with the flow fully developed
(iii) the problem of flow over a triangular step; the results of the flow, friction factor and heat transfer are presented as additional results, and the flow is assumed to be developed at the entrance.

In all the examples described below, the fluid is assumed to be initially at rest with the temperature throughout the domain equal to the entry temperature. The boundary conditions are applied to the fluid at rest and the developments are studied as a transient problem. The solution is assumed to be converged if the change in non-dimensional values of all the variables is below 0·00001. The steady state solutions are used to evaluate the performance parameters such as the friction factor, Nusselt number, etc. The Nusselt number Nu and friction factor F are evaluated from the following equations:[1]

(local) $$Nu_x = (\mathrm{d}\theta/\mathrm{d}Y)/(\theta_w - \theta_m), \qquad (8a)$$

(average) $$Nu_m = (1/X)\int_0^X Nu_x \mathrm{d}X, \qquad (8b)$$

(local) $$F_x = 2(\mathrm{d}U/\mathrm{d}Y)/Re, \qquad (8c)$$

(average) $$F_m = (1/X)\int_0^X F_x \mathrm{d}X, \qquad (8d)$$

$$\theta_m = (1/D)\int_0^D \theta \mathrm{d}Y. \qquad (8e)$$

All quantities referred to in the present paper are non-dimensional.

(i) Flow through parallel channel with backward step

This particular problem is chosen for the purpose of comparison. The problem statement is given in Figure 1(a). The problem is solved for a Reynolds number of 60. The results for the streamfunction and isobars are plotted in Figure 2. The pressure variation along the bottom wall agrees with the results of Lee,[18] as shown in Figure 2(c). The pressure contours shown in Figure 2(b) are smooth without any wiggles. Hence the results of flow over a backward step suggest that the present method could be used to simulate recirculatory flows.

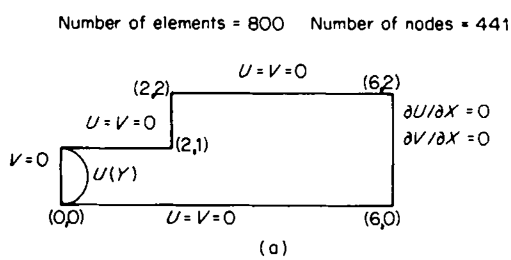


Figure 1(a). Problem of flow over a backward step

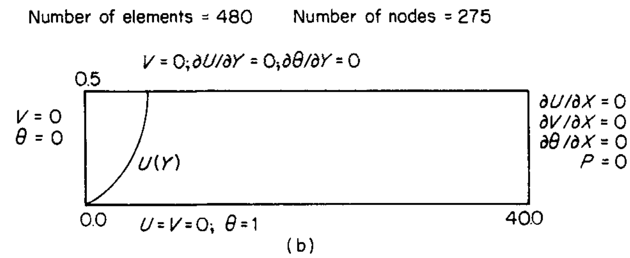


Figure 1(b). Problem of thermal development between parallel plates

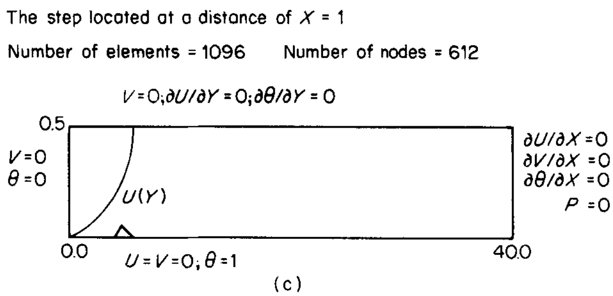


Figure 1(c). Problem of flow over a triangular step

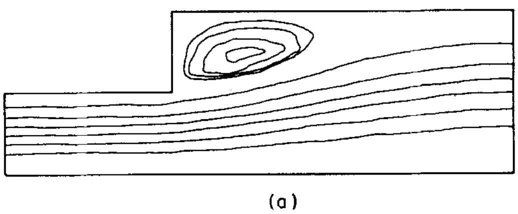


Figure 2(a). Streamlines for flow over a backward step

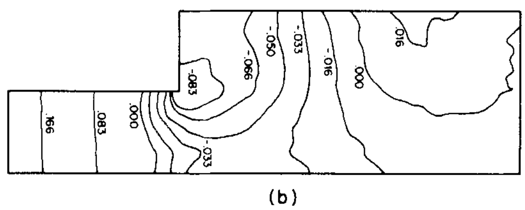


Figure 2(b). Isobar plot for flow over a backward step

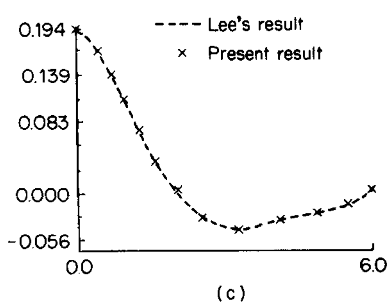


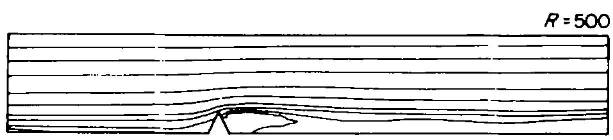
Figure 2(c). Bottom wall pressure variation for flow over a backward step *(ii) Thermal development in flow between parallel plates*

The problem statement is given in Figure 1(b). The problem is the original Graetz problem of thermal development with the velocity profile developed. The flow is allowed to develop from rest and initial temperature equal to inlet temperature. The walls are maintained at a constant temperature. Since the flow is symmetric, one half of the domain is taken for solution. The region is divided into 480 elements having 275 nodes. The Nusselt numbers are calculated from equations (8). The Nusselt numbers evaluated from the calculated values of velocities and temperature agree with the results reported by Shah.[1] The results are tabulated for comparison in Table I.

The solution of cases (i) and (ii) shows that the present method can be used for the analysis of forced convection problems.

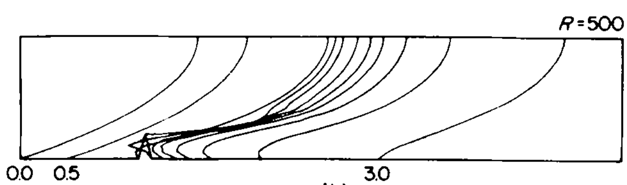
Table I. Mean Nusselt number comparison for thermal development between parallel plates

X^*	Present method	Shah[1]
0·003	13·194	13·420
0·004	12·195	12·248
0·005	11·494	11·413
0·010	9·923	9·891
0·015	9·185	9·107
0·020	8·787	8·716
0·030	8·375	8·324
0·040	8·166	8·128
0·050	8·043	8·011
0·060	7·962	7·933
0·070	7·901	7·877
0·080	7·855	7·835
0·090	7·819	7·807
0·100	7·793	7·776
0·200	7·669	7·658



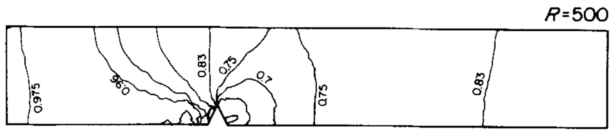
(a)

Figure 3(a). Streamlines for flow over a triangular step



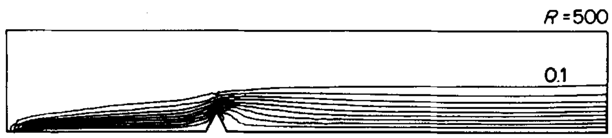
(b)

Figure 3(b). Velocity profiles for flow over a triangular step



(c)

Figure 3(c). Isobar plot for flow over a triangular step



(d)

Figure 3(d). Isotherm plot for flow over a triangular step

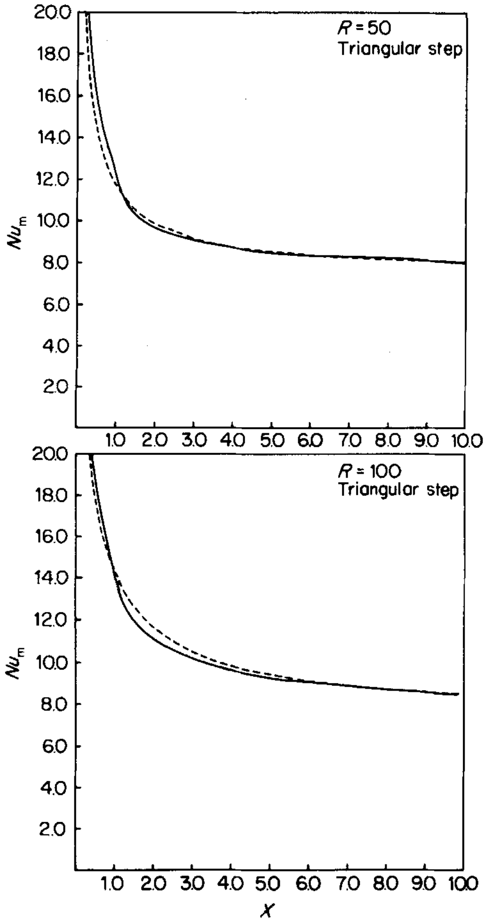


Figure 4(a). Mean Nusselt number variation along the length

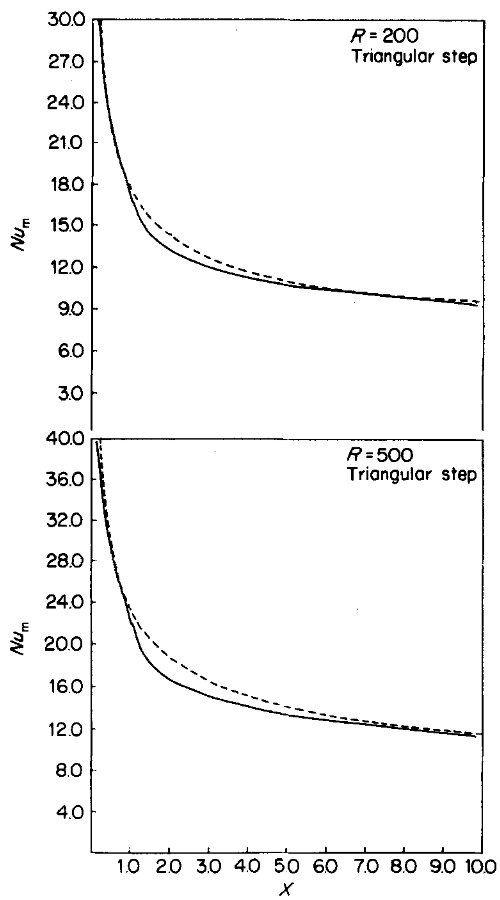


Figure 4(b). Mean Nusselt number variation along the length

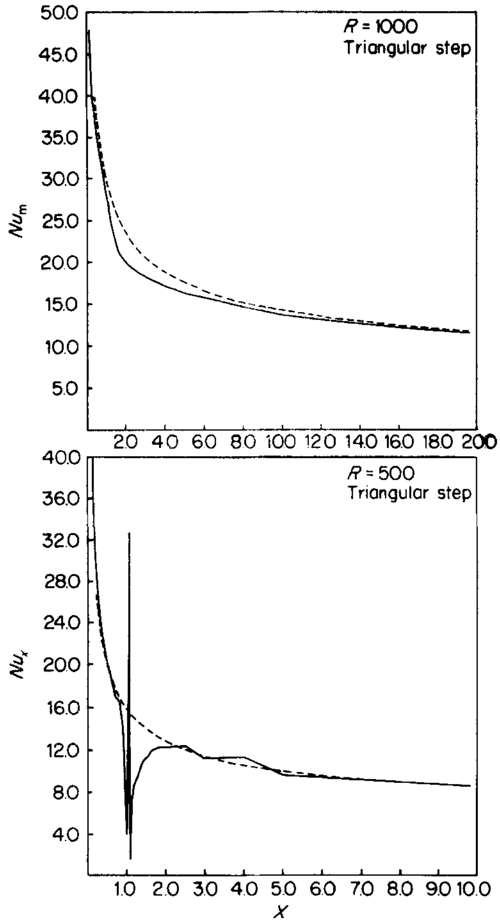


Figure 4(c). Mean and local Nusselt number variation along the length

(iii) Flow over a triangular step

The statement of the problem is given in Figure 1(c). The present paper is devoted to the study of a triangular step which is shorter in length and height than the duct spacing. This is closer to the study of surface roughness effects on the flow and thermal characteristics of ducts.

In the present paper a triangular step of height 0·1 times the duct spacing and base width 0·1 times the duct spacing is considered. The step is located at a distance of one duct width from the entrance. The problem is solved for the Reynolds numbers 50, 100, 200, 500 and 1000. The streamfunction, isobars, isotherms and velocity profiles at different locations along the flow length are shown for a Reynolds number of 500 in Figure 3. The recirculation is smoothly predicted by the present method. The isobar plot does not show any wiggles as reported by previous researchers. The Nusselt numbers and friction factors are evaluated from equations (8). The thermal and flow performances are compared with the simple parallel plate results. The overall pressure drop shows an increase, whereas the mean Nusselt number is less than that of the parallel plates. The mean Nusselt numbers (Nu_m) along the length for various Reynolds numbers are plotted in Figure 4. The simple parallel plate solutions are also plotted as broken lines for comparison. The general characteristics of the variation can be described as follows. The Nusselt number for the step is greater than for the parallel plates up to the beginning of the step. It falls below the parallel plate value and remains lower up to a distance of $X = 10$. It approaches the parallel plate value at a large distance from the entrance. The reduction in Nusselt number increases with increasing Reynolds number. The reason for this is evident from the isotherm plot given in Figure 3(d). In the entrance region the isotherms are closer to the wall. The clustering of isotherms accounts for the high temperature derivative (high flux); hence the Nusselt number is higher near the entrance. As the flow proceeds towards the step, the isotherms climb towards the top of the step, moving away from the wall. The temperature derivative falls as the flow approaches the step; hence the local Nusselt number decreases, as shown in Figure 4(c). This fall in local Nusselt number reduces the average Nusselt number. The isotherms converge on top of the step. Hence the local Nusselt number shows an abrupt increase (Figure 4(c)). The isotherms again become detached from the wall after the step. The decrease in the temperature derivative (flux) after the step has the adverse effect of reducing the average Nusselt number below the parallel plate value. The average Nusselt number can be maintained higher than the parallel plate value for a considerable distance away from the step if the sudden detachment of the isotherms from the wall can be avoided.

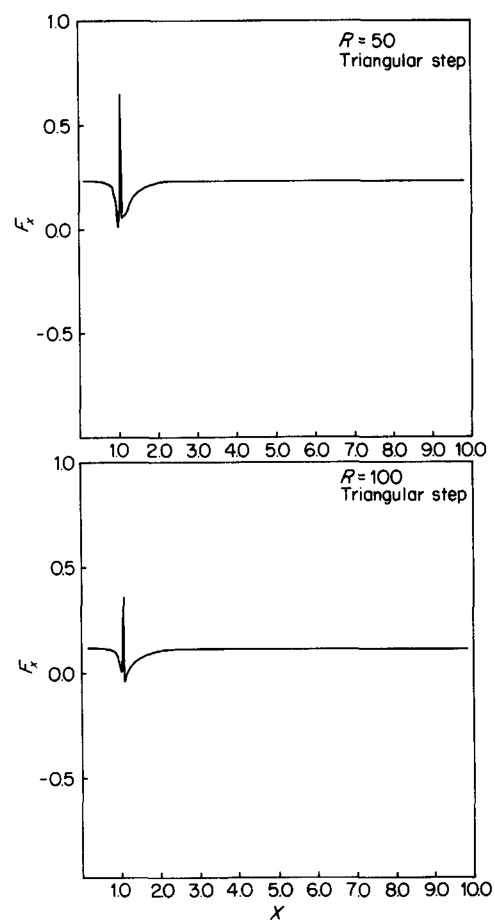


Figure 5(a). Local skin friction variation along the length

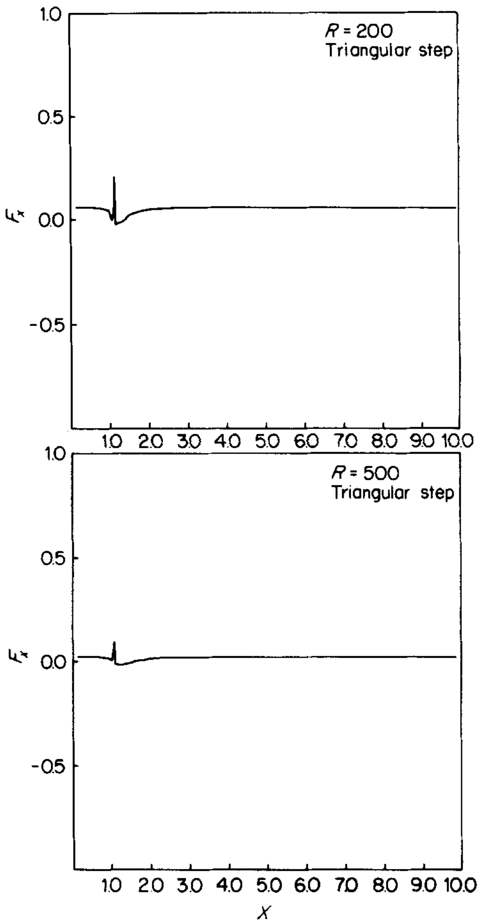


Figure 5(b). Local skin friction variation along the length

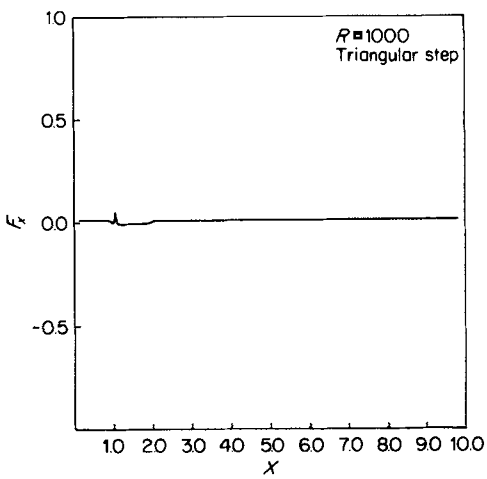


Figure 5(c). Local skin friction variation along the length

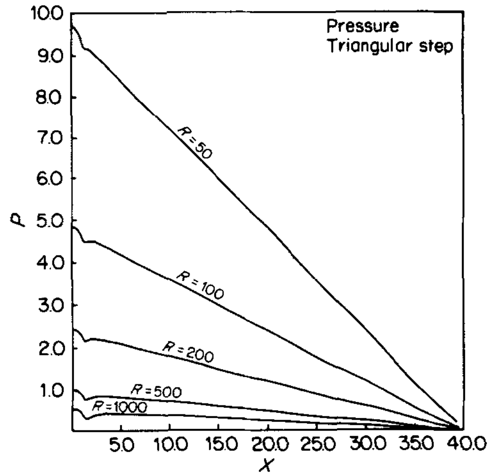


Figure 6. Pressure variation along the centreline for flow over a square step

The skin friction factor variation along the length is shown in Figure 5. The recirculation region is clearly evident from the change in the sign of the friction factor. The total pressure drop for the flow over the step is more than for the parallel plates. Plots of the centreline pressure variation for different Reynolds numbers are given in Figure 6. The presence of the step increases the pressure drop near the step. This is evident from the sudden drop in pressure near the step zone in Figure 6. The effect of the step on the pressure drop increases with increasing Reynolds number.

CONCLUSIONS

The application of the velocity correction method to forced convection problems is demonstrated with examples. The method is capable of producing wiggle-free solutions for forced convection problems. Upwinding is totally avoided in the present method. The velocities and pressure are interpolated to equal order. The problem of flow over a triangular step of dimensions shorter than the duct width is solved. The flow and thermal results are presented in terms of Nusselt numbers and friction factors which are relevant to heat exchanger design.

APPENDIX: NOMENCLATURE

c	heat capacity
D	duct spacing
F	friction factor
k	conductivity
N	shape functions
Nu	Nusselt number
p	dimensional pressure
P	non-dimensional pressure
Pr	Prandtl number
Re	Reynolds number
t	dimensional time
T	dimensional temperature

u	dimensional x-component velocity
U	non-dimensional x-component velocity
v	dimensional y-component velocity
V	non-dimensional y-component velocity
x	dimensional distance in x-direction
X	non-dimensional distance in x-direction
X^*	non-dimensional distance, $X^* = X/4Re$
y	dimensional distance in y-direction
Y	non-dimensional distance in y-direction

Subscripts

e	entry values
o	exit values
m	mean values
w	refers to wall
n	refers to quantities at time t_n
x	local values

Greek symbols

τ	non-dimensional time
θ	non-dimensional temperature
μ	kinematic viscosity
ρ	density

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